Rowmotion on interval-closed sets

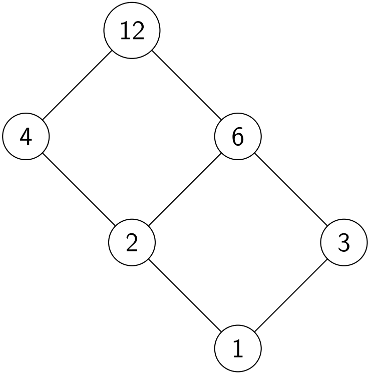
Jennifer Elder, Nadia Lafrenière, Erin McNicholas,
Jessica Striker, Amanda Welch

CAAC 2024
UQAM, January 26, 2024

Posets

A *poset* is a partially ordered set. Here, we study finite posets.

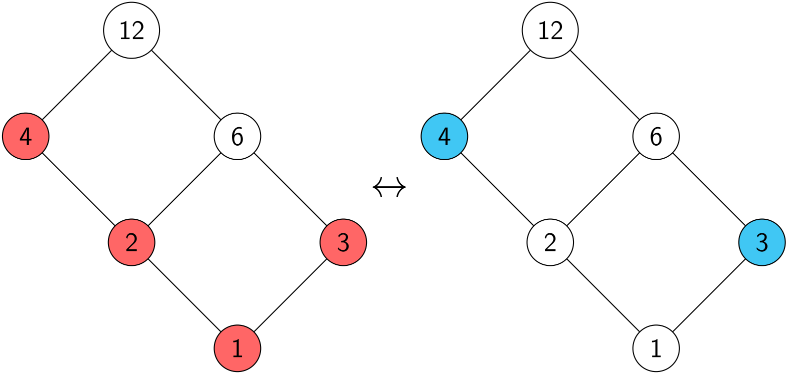
We draw posets using their *cover relations*, that appear in a directed graph, called the *Hasse diagram*.



The divisors lattice

Order ideals

An *order ideal* of a poset is a subset I such that if x in I and $y \leq x$, then $y \in I$.



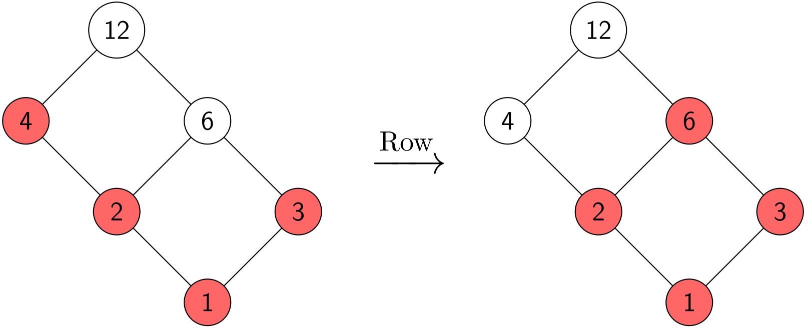
Proposition

There is a bijection between order ideals and antichains.

Rowmotion: classical version

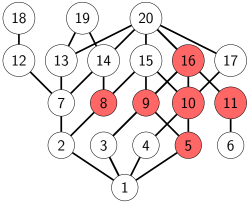
Rowmotion is a bijection on order ideals of posets.

- Global action: order ideal spanned by the minima of the complement
- Local action: toggling top-to-bottom

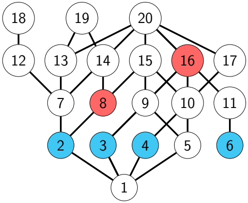


Interval-closed sets

An *interval-closed set* of a poset P is a subset I of P such that for all $x, y \in I$, $z \in I$ for all $x \leq z \leq y$.
Order ideals and order filters are interval-closed sets, but there are many more.



$\leftrightarrow$

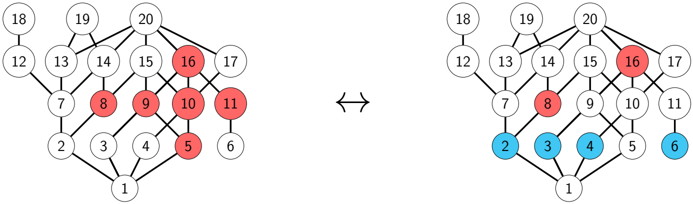


Proposition

There is a bijection between interval-closed sets and pairs of antichains such that one is fully below the other.

Interval-closed sets

An *interval-closed set* of a poset P is a subset I of P such that for all $x, y \in I$, $z \in I$ for all $x \leq z \leq y$.
Order ideals and order filters are interval-closed sets, but there are many more.

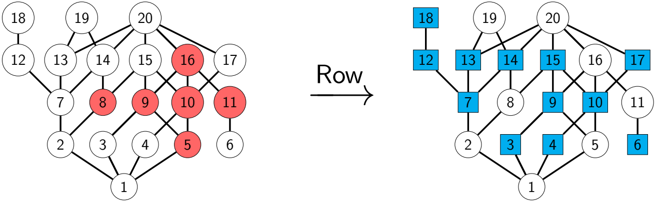


Problem (Open, for the most part)

How many interval-closed sets does a poset have?

Rowmotion on interval-closed sets

We define rowmotion on ICS by toggling from top to bottom:



Problem

Is there a global description of rowmotion?

Rowmotion as a global action

Theorem (Elder, L., McNicholas, Striker, Welch)

Rowmotion on an interval-closed set I is described globally as

$$\mathrm{Row}(I) = \mathrm{Inc}(I) \cup \Big(\Delta\mathrm{Inc}_I(\mathrm{Ceil}(I)) - (I \cup \Delta\mathrm{Ceil}(I))\Big)$$
$$\cup \Big(\Delta\mathrm{Ceil}(I) - \Delta(\mathrm{Min}(I) \cap \Delta\mathrm{Ceil}(I))\Big),$$

where $\mathrm{Inc}(I)$ is the set of elements incomparable with I, and the ceiling is the set of the minimal elements not in I and above elements of I.
$\Delta(I)$ is the order ideal spanned by I.

Rowmotion as a global action

Theorem (Elder, L., McNicholas, Striker, Welch)

Rowmotion on an interval-closed set I is described globally as

$$\mathrm{Row}(I) = \mathrm{Inc}(I) \cup \Big(\Delta\mathrm{Inc}_I(\mathrm{Ceil}(I)) - (I \cup \Delta\mathrm{Ceil}(I))\Big) \cup \Big(\Delta\mathrm{Ceil}(I) - \Delta(\mathrm{Min}(I) \cap \Delta\mathrm{Ceil}(I))\Big),$$

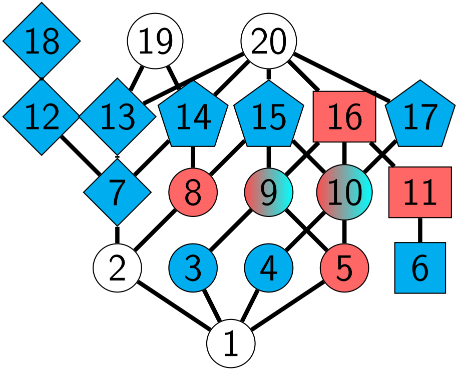
where $\mathrm{Inc}(I)$ is the set of elements incomparable with I, and the ceiling is the set of the minimal elements not in I and above elements of I.
$\Delta(I)$ is the order ideal spanned by I.

This alternate definition speeds up the computations a lot!

Rowmotion as a global action

Example:

$$\mathrm{Row}(I) = \mathrm{Inc}(I) \cup \Big(\Delta\mathrm{Inc}_I(\mathrm{Ceil}(I)) - (I \cup \Delta\mathrm{Ceil}(I))\Big)$$
$$\cup \Big(\Delta\mathrm{Ceil}(I) - \Delta(\mathrm{Min}(I) \cap \Delta\mathrm{Ceil}(I))\Big),$$



red $= I$

blue $= \mathrm{Row}(I)$

◇ Incomparable with I

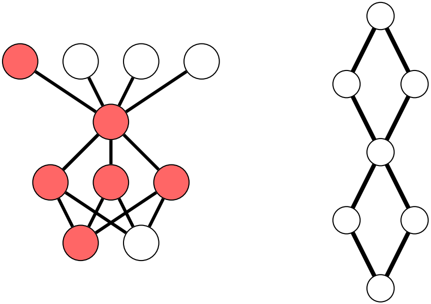
□ Incomparable with ⬠

⬠ Ceiling

Ordinal sums of antichains

Definition

Given two antichains $\mathbf{a_1}$ and $\mathbf{a_2}$, their *ordinal sum* $\mathbf{a_1} \oplus \mathbf{a_2}$ is the poset in which $x \leq y$ for all pair $(x, y) \in \mathbf{a_1} \times \mathbf{a_2}$.

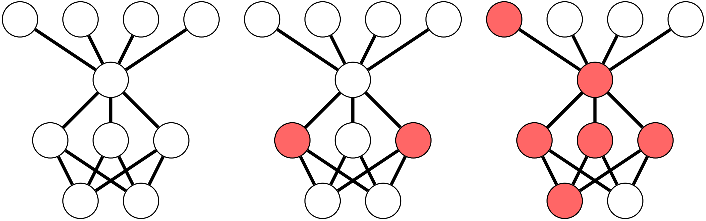


$\mathbf{2} \oplus \mathbf{3} \oplus \mathbf{1} \oplus \mathbf{4}$ and the *stacked diamond* $D(5, 2) = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{1}$.

Enumeration of ICS on ordinal sums of antichains

Theorem (Elder, L., McNicholas, Striker, Welch)

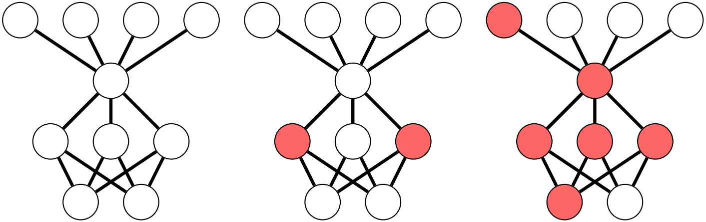
The number of ICS in $\mathbf{a_1} \oplus \mathbf{a_2} \oplus \ldots \oplus \mathbf{a_n}$ *is*
$1 + \sum_{1 \leq i \leq n}(2^{a_i} - 1) + \sum_{1 \leq i < j \leq n}(2^{a_i} - 1)(2^{a_j} - 1)$.



Enumeration of ICS on ordinal sums of antichains

Theorem (Elder, L., McNicholas, Striker, Welch)

The number of ICS in $\mathbf{a_1} \oplus \mathbf{a_2} \oplus \ldots \oplus \mathbf{a_n}$ *is*
$1 + \sum_{1 \leq i \leq n}(2^{a_i} - 1) + \sum_{1 \leq i < j \leq n}(2^{a_i} - 1)(2^{a_j} - 1)$.



$+$ more specific results for:

- repeated antichains of the same size
- altered stacked diamonds

Rowmotion orbits on ordinal sums of antichains

Theorem (Elder, L., McNicholas, Striker, Welch)

Let $n \geq 3$. The interval-closed sets of $\mathbf{a_1} \oplus \mathbf{a_2} \oplus \ldots \oplus \mathbf{a_n}$ have rowmotion order $2n(n+2)$ when n is odd and $n(n+2)/2$ when n is even.

$+$ We know a complete description of its rowmotion orbit structure.

Rowmotion as a global action on ordinal sums of antichains

Theorem (Elder, L., McNicholas, Striker, Welch)

Let P be an ordinal sum of antichains. Then rowmotion on the ICS I is either

- *the complement of I, if I is either empty, contains all of the top antichain, or is a subset of the top antichain.*
- *the difference of the order ideals spanned by the ceiling of I and the minima of I.*

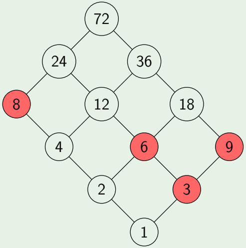
Products of chains

Definition

The *divisors lattice* of n is the poset containing all the divisors of n, in which $a \leq b$ if a divides b.
If n admits a prime factor decomposition $p_1^{a_1} \ldots p_k^{a_k}$, the divisors lattice is equivalent to the product of chains $[a_1 + 1] \times \ldots \times [a_k + 1]$.

Example

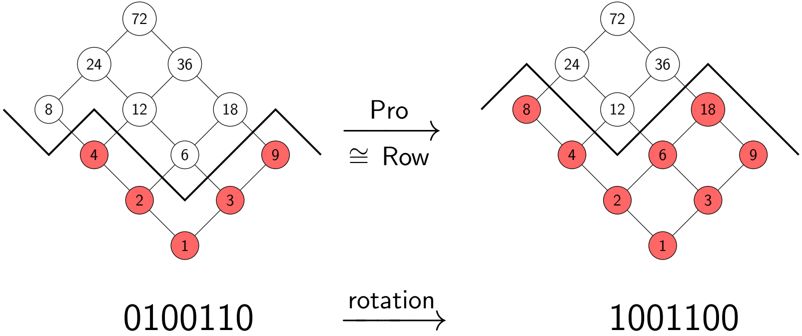
The divisors lattice of $72 = 2^3 \cdot 3^2$ is $[3] \times [4]$.



“Classical” (order ideal) rowmotion on product of chains

Order-ideal rowmotion on products of two chains $[a] \times [b]$ has order $a + b$. It has the same orbit structure as the promotion operation.

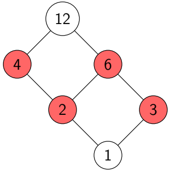
Promotion is equivalent to rotation of a word of length $a + b$.

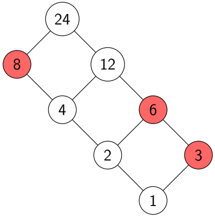


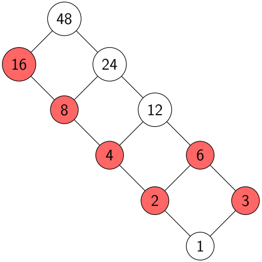
ICS-rowmotion on products of chains

Problem (Open)

What is the order of rowmotion? The orbit structure?







Order	60	210	1144
Orbits	$[2^2, 3^1, 6^2, 20^1]$	$[2^1, 5^3, 6^1, 7^4, 10^2]$	$[2^1, 4^2, 8^7, 26^1, 44^1]$

Enumeration of ICS on products of chains

Problem

How many ICS are there in the product of chains $[a] \times [b]$?

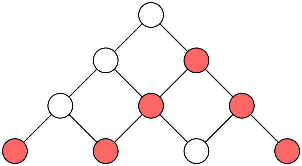
Theorem (Elder, L., McNicholas, Striker, Welch)

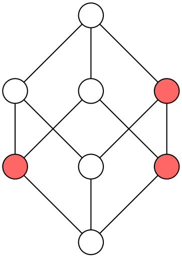
- *The number of ICS of $[2] \times [n]$ is* $1 + 2\left(\binom{n}{2} + n\right) + \frac{n+1}{2}\binom{n+2}{3}$.
- *The number of interval-closed sets of $[a] \times [b]$ with at least one element in each of the a chains is in bijection with the order ideals of $[a] \times [b-1] \times [2]$, and is thus counted by the Narayana number $N(a+b, b)$.*

Joel Brewster Lewis found a generating function for the number of ICS of $[a] \times [b]$.

Further questions: other posets

There are plenty of other posets that we have not investigated:





The number of interval-closed sets is A367316 for the type A_n root poset and A369313 for the boolean lattice of dimension n.

Further questions and open problems

Enumeration problems:

- Count the number of ICS for various posets

Dynamics problems:

- Give orbit structure of rowmotion for each of these posets
- Homomesic statistics? We got conjectures in our paper on #max-#min and signed cardinality, and we proved some results regarding #max-#min and toggleability.

Resources

Interested in working on these problems? We got resources!

Toggling, rowmotion and homomesy on interval-closed sets

ArXiv:2307.08520; To appear in *Journal of Combinatorics*

OEIS sequences A001263, A367109 A367316 and A369313

FindStat statistic 1909: The number of interval-closed sets of a poset

A worksheet with all our SageMath code is available on CoCalc (link in the paper)

Thank you!